Contributions to Political Economy (2021) **40**, 1–25

COMPETING THEORIES OF VALUE: A SPECTRAL ANALYSIS

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This paper is based on a spectral–controllability analysis of the Sraffian price system and develops a relevant reconstruction of the theory of value. It shows that (i) the hitherto competing theories of value correspond to specific complex plane locations of the eigenvalues of the vertically integrated technical coefficients matrix; and (ii) the real-world economies cannot be coherently analyzed in terms of the traditional theory of value (classical, Marxian, Austrian, and neoclassical), despite the fact that they are characterized by rather low degrees and relatively low numerical ranks of price controllability. Hence, on the one hand, the Sraffian theory of value is not only the most general to date, but also empirically relevant. On the other hand, the real-world economies constitute almost uncontrollable systems, and this explains, in turn, the specific shape features of the empirical price–wage–profit rate curves that are at the heart of the capital theory debate.

JEL Classification: B51; B53; C67; D46; D57

I. INTRODUCTION

In spite of their fundamental conceptual differences, the theories of value of the traditional political economy (classical, Marxian, Austrian, and neoclassical) reduce, in essence, to the existence of an unambiguous relationship between, on the one hand, the movement of the long-period relative price of two commodities arising from changes in income distribution and, on the other hand, the difference in the capital intensities of the industries producing these commodities. Since Sraffa's (1960) contribution, it has been gradually recognized, however, that such a relationship does not necessarily exist: Even in a world of fixed input–output coefficients and at least three commodities, produced by means of themselves and homogeneous labor, long-period relative prices can change in a complicated way as income distribution changes. This phenomenon has crucial implications for all the traditional theory of value, capital, distribution–growth, and international trade, while its investigation led to

*I am indebted to an anonymous referee and Murray Milgate for critical and constructive comments and suggestions. Various versions of this paper were presented at workshops of the 'Study Group on Sraffian Economics' and lectures at the Panteion University (May 2016–November 2019): I am grateful to the participants and, in particular, to Despoina Kesperi, Nikolaos Rodousakis, George Soklis, and Panagiotis Veltsistas for very helpful remarks and stimulating discussions.

the formation of a *new* theory of value, namely, the 'Sraffian theory of value,' which includes the abovementioned relationship between price variation and capital intensity difference as its special or limiting case.

Following a post-2000 line of research that couples Piero Sraffa's theory of value with Rudolf Emil Kalman's (1960) general theory of control systems, this paper develops a unified treatment of the problem of value–capital on both theoretical and empirical grounds.[1] In particular, it shows that the hitherto competing theories of value correspond to specific complex plane locations of the eigenvalues of the vertically integrated technical coefficients matrix, identifies new aspects of the Sraffian theory of value, by generalizing Sraffa's (1960, pp. 36–38) (in)famous 'old wine–oak chest' example, and, finally, zeroes in on the spectral 'imprint' of actual price–wage–profit rate systems by detecting the singular value configuration of the relevant Krylov–controllability matrices. Hence, it (i) points out a spectral reduction or reconstruction of the theory of value; (ii) highlights the general importance, both theoretical and applied, of the Sraffian theory of value; and (iii) supports the recently proposed link between the tendency of actual economies to respond as uncontrollable systems and the specific shape features of the empirical price–wage–profit rate curves (Mariolis and Tsoulfidis, 2018). *If* the key issue is 'to decide whether the real world is nearer to the idealized polar cases represented by (a) the neoclassical parable or (b) the simple reswitching paradigm' (Samuelson, 1980, p. 576),[2] then our analysis and findings suggest that the real world is far from the former case and, therefore, close to the latter case.

The remainder of the paper is structured as follows. Section II analyzes the Sraffian price system and determines the complex plane location of the hitherto competing theories of value. Section III provides evidence indicating the empirical relevance of the Sraffian theory of value and, at the same time, the almost uncontrollability of actual economies. Finally, Section IV concludes the paper.

II. COMPETING THEORIES OF VALUE AND NON-DOMINANT EIGENVALUES

II. I.a The long-period price system

Consider a closed, linear and viable economy involving only single products, 'basic' commodities (in the sense of Sraffa, 1960, pp. 7–8), circulating capital and homogeneous labor. Furthermore, assume that: (i) wages are paid at the end of the common production period; (ii) the matrix of direct technical coefficients is diagonalizable; and (iii) the price of a commodity obtained as an output at the end of the production period is the same as the price of that commodity used as an input at the beginning of that

[1] The seminal papers in this field are Mariolis (2003); Mariolis and Tsoulfidis (2009).

[2] Cited by Cohen (1993, p. 155). Zambelli (2004, p. 107, footnote 2) aptly observes that re-switching is a sub-case of the—much more general—*non*-neoclassical case.

period ('stationary prices'). Hence, the price side of the economy is described by[3]

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{l}^{\mathrm{T}} + (1+r)\,\mathbf{p}^{\mathrm{T}}\mathbf{A} \tag{1}$$

where $\mathbf{p}^{\mathrm{T}}$ denotes a $1xn$ vector of production prices, w is the money wage rate, r is the uniform profit rate, $\mathbf{l}^{\mathrm{T}}$ $(> \mathbf{0}^{\mathrm{T}})$ is the $1xn$ vector of direct labor coefficients, and $\mathbf{A}$ is the irreducible nxn matrix of direct technical coefficients, with $\lambda_{\mathbf{A}1} < 1$.

After rearrangement, equation (1) becomes

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{v}^{\mathrm{T}} + \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} \tag{2}$$

where $\mathbf{v}^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I}-\mathbf{A}]^{-1}$ $(> \mathbf{0}^{\mathrm{T}})$ denotes the vector of vertically integrated labor coefficients, or the traditionally so-called 'labor values,' and $\mathbf{H} \equiv \mathbf{A}[\mathbf{I}-\mathbf{A}]^{-1}$ $(> \mathbf{0})$ the vertically integrated technical coefficients matrix. Moreover, $\rho \equiv rR^{-1}, 0 \leq \rho \leq 1$, denotes the relative profit rate, which equals the share of profits in the Sraffian Standard system, and $R \equiv \lambda_{\mathbf{A}1}^{-1} - 1 = \lambda_{\mathbf{H}1}^{-1}$ the maximum possible profit rate (i.e. the profit rate corresponding to $w = 0$ and $\mathbf{p} > \mathbf{0}$), which equals the ratio of the net product to the means of production in that system (Sraffa, 1960, pp. 21–23). Finally, $\mathbf{J} \equiv R\mathbf{H}$ denotes the normalized vertically integrated technical coefficients matrix, $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$, and the moduli of the normalized non-dominant eigenvalues of system (2) are less than those of system (1), i.e. $|\lambda_{\mathbf{J}k}| < |\lambda_{\mathbf{A}k}|\lambda_{\mathbf{A}1}^{-1}$ holds for all k (see, e.g. Mariolis and Tsoulfidis, 2016a, p. 22).

If Sraffa's Standard commodity is chosen as the *numéraire*, i.e. $\mathbf{p}^{\mathrm{T}}\mathbf{z} = 1$, where $\mathbf{z} \equiv [\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1}$ and $\mathbf{l}^{\mathrm{T}}\mathbf{x}_{\mathbf{A}1} = 1$, then equation (1) implies that the 'wage–relative profit rate curve' is the following linear relation:

$$w = 1 - \rho \tag{3}$$

with $w(0) = 1$ and $w(1) = 0$. Substituting equation (3) into equation (2) yields

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\,\mathbf{v}^{\mathrm{T}} + \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} \tag{4}$$

or, if $\rho < 1$,

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\,\mathbf{v}^{\mathrm{T}}[\mathbf{I}-\rho\mathbf{J}]^{-1} = (1-\rho)\,\mathbf{v}^{\mathrm{T}}\left[\mathbf{I} + \rho\mathbf{J} + \rho^2\mathbf{J}^2 + \rho^3\mathbf{J}^3 + \ldots\right] \tag{5}$$

[3] The transpose of an $n \times 1$ vector $\mathbf{x}$ is denoted by $\mathbf{x}^{\mathrm{T}}$, and the diagonal matrix formed from the elements of $\mathbf{x}$ will be denoted by $\hat{\mathbf{x}}$. Furthermore, $\lambda_{\mathbf{A}1}$ will denote the Perron–Frobenius (P–F) eigenvalue of a semi-positive $n \times n$ matrix $\mathbf{A} \equiv [a_{ij}]$, and $(\mathbf{x}_{\mathbf{A}1}, \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}})$ the corresponding eigenvectors, while $\lambda_{\mathbf{A}k}$, $k = 2, \ldots, n$ and $|\lambda_{\mathbf{A}2}| \geq |\lambda_{\mathbf{A}3}| \geq \cdots \geq |\lambda_{\mathbf{A}n}|$, will denote the non-dominant eigenvalues, and $(\mathbf{x}_{\mathbf{A}k}, \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}})$ the corresponding eigenvectors. Finally, $\mathbf{e}$ will denote the summation vector, i.e. $\mathbf{e} \equiv [1, 1, \ldots, 1]^{\mathrm{T}}$, $\mathbf{e}_i$ the ith unit vector, and $\mathbf{I}$ the nxn identity matrix.

which gives the commodity prices, expressed in terms of Sraffa's Standard commodity, as polynomial functions of ρ.

Equation (4) indicates that p_j is a convex combination of v_j and $\mathbf{p}^{\mathrm{T}}\mathbf{J}\mathbf{e}_j$, where the latter equals the ratio of means of production in the vertically integrated industry producing commodity j to means of production in the Sraffian Standard system. From this equation it follows that $\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}}$ and $\mathbf{p}^{\mathrm{T}}(1)$ is the left P–F eigenvector of $\mathbf{J}$, expressed in terms of Sraffa's Standard commodity, i.e.

$$\mathbf{p}^{\mathrm{T}}(1) = \left(\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{z}\right)^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} = \left(\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\left[\mathbf{I}-\mathbf{A}\right]\mathbf{x}_{\mathbf{A}1}\right)^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$$

or, since $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{A}1} = (1-\lambda_{\mathbf{A}1})\mathbf{x}_{\mathbf{A}1}$ and matrices $\mathbf{A}$ and $\mathbf{J}$ have the same set of eigenvectors,

$$\mathbf{p}^{\mathrm{T}}(1) = \left[(1-\lambda_{\mathbf{A}1})\,\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}\right]^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} \tag{6}$$

Equation (5) is the reduction of commodity prices to 'dated quantities' of normalized vertically integrated labor in terms of Sraffa's Standard commodity. In the general case, therefore, commodity prices are ratios of polynomials of degree $n-1$ in ρ and, therefore, may admit up to $2n-4$ extreme points.

Finally, it should be added that:[4] (i) Non-diagonalizable systems are of measure zero in the set of all systems and, hence, not generic, while given any $\mathbf{A}$ and an arbitrary $\varepsilon \neq 0$, it is possible to perturb the entries of $\mathbf{A}$ by an amount less than $|\varepsilon|$ so that the resulting matrix is diagonalizable. (ii) These fundamental price relationships remain valid for the cases of (a) fixed capital *à la* Leontief–Bródy; and (b) differential profit and wage rates (provided that these variables exhibit a stable structure in relative terms). For instance, in the former case, $\mathbf{v}^{\mathrm{T}}$ and $\mathbf{H}$ should be replaced by $\mathbf{l}^{\mathrm{T}}[\mathbf{I}-(\mathbf{A}+\mathbf{A}^{\mathrm{D}})]^{-1}$ and $\mathbf{A}^{\mathrm{C}}[\mathbf{I}-(\mathbf{A}+\mathbf{A}^{\mathrm{D}})]^{-1}$, respectively, where $\mathbf{A}^{\mathrm{D}}$ denotes the matrix of depreciation coefficients, and $\mathbf{A}^{\mathrm{C}}$ is the matrix of capital stock coefficients. Nevertheless, the said price relationships do not necessarily remain valid for joint production economies.

II. I.b Controllable and almost uncontrollable price systems

Consider the following dynamic version of the price system (2):

$$\mathbf{p}_{t+1}^{\mathrm{T}} = w_{t+1}\mathbf{v}^{\mathrm{T}} + \rho^{\mathrm{n}}\mathbf{p}_t^{\mathrm{T}}\mathbf{J}, t = 0, 1, \ldots$$

where ρ^{n} denotes the exogenously given nominal relative profit rate, and $[w_0 = 0, \mathbf{p}_0^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}]$ (consider Solow, 1959). This dynamic system is said to be 'completely controllable' or 'controllable of rank n' (Kalman, 1960) if the initial state $\mathbf{p}_0^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$ can be transferred, by application of w_t, to any state, in a finite length of time. Thus,

[4] See Aruka (1991, pp. 74–76); Kurz and Salvadori (1995, Chaps. 7, 8, and 11); Mariolis and Tsoulfidis (2016a, pp. 22–32); Sraffa (1960, Part 2).

as Luenberger (1979) remarks, complete controllability corresponds directly to the intuitive notion of being able to control the system state (p. 277).

The said system is completely controllable iff the $n \times n$ Krylov–controllability matrix, i.e. the matrix of the first n dated quantities of normalized vertically integrated labor (see equation (5)),

$$\mathbf{K} \equiv \left[\mathbf{p}(0), \mathbf{J}^{\mathrm{T}}\mathbf{p}(0), \ldots, \left(\mathbf{J}^{\mathrm{T}}\right)^{n-1}\mathbf{p}(0)\right]^{\mathrm{T}}$$

has rank equal to n or, equivalently, iff $\mathbf{p}^{\mathrm{T}}(0)$ is not orthogonal to any (real or complex) right eigenvector of $\mathbf{J}$.[5] By contrast, iff rank$[\mathbf{K}] = m < n$, then the system is said to be 'uncontrollable' or, more specifically, 'controllable of rank m'. Iff the dimension of an eigenspace associated with an eigenvalue of $\mathbf{J}$ is larger than 1 or, equivalently, iff $\mathbf{J}$ satisfies a polynomial equation of degree less than n, then the system is uncontrollable *whatever* $\mathbf{p}^{\mathrm{T}}(0)$ is (Ford and Johnson, 1968).[6]

Furthermore, it can be proved that, when the dynamic system is completely controllable, the stationary price vectors relative to any n distinct values of the profit rate $(0 \le \rho \le 1)$ are linearly independent (Schefold, 1976; Bidard and Salvadori, 1995; Kurz and Salvadori, 1995, Chap. 6). Therefore, the curve $\mathbf{p}^{\mathrm{T}}(\rho)$ is entirely contained in a space of dimension n, and cuts any space of dimension $n'(< n)$ in no more than n' points. Hence, the price movement arising from changes in income distribution can be characterized as—somewhat—'erratic.' By contrast, when the dynamic system is controllable of rank $m(< n)$, the stationary price vectors relative to any $m+1$ distinct values of the profit rate are linearly dependent and, therefore, the curve $\mathbf{p}^{\mathrm{T}}(\rho)$ is entirely contained in a space of dimension m. In that case, there are n-m vectors $\mathbf{z}'$ such that $\mathbf{K}\mathbf{z}' = \mathbf{0}$ and, therefore, $\mathbf{p}^{\mathrm{T}}(\rho)\mathbf{z}' = 0$ (see equation (5)). Hence, a change of *numéraire*, from $\mathbf{z}$ to $\mathbf{z} + \alpha\mathbf{z}'$, where α is a given scalar, has no effect on the wage–relative profit rate curve (and on the stationary prices; Miyao, 1977). In a word, then, we have the following pairs of 'opposites': controllability (uncontrollability) implies unpredictable (predictable) stationary price and wage movements arising from changes in the profit rate. And predictability decreases with increasing rank of controllability.[7]

This approach provides only a *yes/no* criterion for complete controllability, while uncontrollable systems are of measure zero in the set of all systems and, thus, not

[5] The rank–eigenvector conditions for controllability are known as 'Popov (1966)–Belevitch (1968)–Hautus (1969) tests or criteria.'

[6] For various versions of the concept of controllability, and its relevance to economic analysis and policy, see Aoki (1976); Hansen and Sargent (2008); McFadden (1969); Wohltmann (1985); and the review paper by Chatelain and Ralf (2020). In the 1970s, Schefold (1971, 1976) introduced the following concept of 'regularity' of a production technique: A production technique $[\mathbf{A}, \mathbf{l}^{\mathrm{T}}]$ is said to be 'regular' iff (i) $\mathbf{l}^{\mathrm{T}}$ is not orthogonal to any right eigenvector of $\mathbf{A}$; and (ii) there exists exactly one right eigenvector (up to a factor) associated with each eigenvalue of $\mathbf{A}$. Equivalently, iff the matrix $[\mathbf{l}, \mathbf{A}^{\mathrm{T}}\mathbf{l}, \ldots, (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{l}]^{\mathrm{T}}$ has rank equal to n. Condition (ii) is superfluous.

[7] For a detailed restatement of fundamental structural features of Sraffa's theory of value in terms of Kalman's general theory of control systems, see Mariolis et al. (2021, Chap. 2).

generic or, in other words, systems are *almost always* controllable (Kalman *et al.*, 1963; for a recent discussion, see Cowan *et al.*, 2012). However,

> [i]n the *real* world [...] it may not be possible to make such sharp distinctions. [...] The problem with the standard definition of controllability [...] is that it can lead to discontinuous functions of the system parameters: an arbitrarily small change in a system parameter can cause an abrupt change in the rank of the matrix by which controllability [...] is determined. It would be desirable to have definitions which can vary continuously with the parameters of the system and thus can reflect the *degree of controllability* of the system. Kalman *et al*. (1963) recognized the need and suggested using the determinant of the corresponding test matrix [**K**] as a measure of the degree of controllability [...]. Friedland (1975), noting that basing the degree of controllability [...] on the determinant of the test matrix suffers from sensitivity to the scaling of the state variables, suggested using the ratio of the smallest of the singular values to the largest as a preferable measure. (Friedland, 1986, p. 220; emphasis added)

In this connection, therefore, matrix **J** can be decomposed as ('spectral decomposition'; see, e.g. Meyer, 2001, 517–518)

$$\mathbf{J} = \left(\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}\right)^{-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} + \sum_{k=2}^{n}\lambda_{\mathbf{J}k}\left(\mathbf{y}_{\mathbf{J}k}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}k}\right)^{-1}\mathbf{x}_{\mathbf{J}k}\mathbf{y}_{\mathbf{J}k}^{\mathrm{T}} \tag{7}$$

or $\mathbf{J} = \mathbf{X}_{\mathbf{J}}\hat{\boldsymbol{\lambda}}_{\mathbf{J}}\mathbf{X}_{\mathbf{J}}^{-1}$, where $\mathbf{X}_{\mathbf{J}}$ and the diagonal matrix $\hat{\boldsymbol{\lambda}}_{\mathbf{J}}$ are matrices formed from the right eigenvectors and the eigenvalues of **J**, respectively, while $\mathbf{X}_{\mathbf{J}}^{-1}$ equals the matrix formed from the left eigenvectors of **J**. Equation (7) implies, in turn, that the Krylov matrix can be expressed as a product of three matrices:

$$\mathbf{K} = \mathbf{V}\hat{\boldsymbol{\omega}}\mathbf{X}_{\mathbf{J}}^{-1}$$

where $\mathbf{V} \equiv (\lambda_{\mathbf{J}j})^{i-1}$ denotes the Vandermonde matrix of the eigenvalues of **J**, and $\hat{\boldsymbol{\omega}}$ the diagonal matrix formed from the elements of $\boldsymbol{\omega}^{\mathrm{T}} \equiv \mathbf{p}^{\mathrm{T}}(0)\mathbf{X}_{\mathbf{J}}$. Consequently, the determinant of **K** is given by

$$\det[\mathbf{K}] = \det[\mathbf{V}]\det\left[\hat{\boldsymbol{\omega}}\right]\det\left[\mathbf{X}_{\mathbf{J}}^{-1}\right] \tag{8}$$

where $\det[\mathbf{V}] = \prod_{1\le i<j\le n}(\lambda_{\mathbf{J}j} - \lambda_{\mathbf{J}i})$. Finally, the 'degree of controllability' is defined as

$$DC \equiv \sigma_{\mathbf{K}n}\sigma_{\mathbf{K}1}^{-1} \tag{9}$$

where $0 \leq DC < 1$, and $\sigma_{\mathbf{K}1}, \sigma_{\mathbf{K}n}$ denote the largest and the smallest singular values of $\mathbf{K}$, respectively, while DC^{-1} equals the 'condition number' of $\mathbf{K}$. When $DC = 0$, the economy is uncontrollable; otherwise, it is completely controllable. Nevertheless, when the value of DC is 'very small', the controllability is 'weak' or 'poor'; in other words, the economy is said to be 'almost uncontrollable.'[8]

II. I.c Theories of value

As is well known, in the Ricardo–Marx–Dmitriev–Samuelson 'equal value compositions of capital' case, $\mathbf{l}^{\mathrm{T}}$ ($\mathbf{v}^{\mathrm{T}}$) is the left P–F eigenvector of $\mathbf{A}$ (of $\mathbf{J}$). Therefore, commodity prices are independent of income distribution, and equal to the labor values, i.e. $\mathbf{p}^{\mathrm{T}}(\rho) = \mathbf{p}^{\mathrm{T}}(0) = \mathbf{p}^{\mathrm{T}}(1)$, or, in other words, the 'labor theory of value' appears to hold true. In that case, the economy is controllable of rank 1 irrespective of the rank of $\mathbf{J}$.

In the two-industry case, the functions $p_j(\rho)$ are necessarily monotonic and, therefore, the direction of relative price movement is governed only by the differences in the relevant capital intensities ('capital-intensity effect'; see Kurz and Salvadori, 1995, Chap. 3; Pasinetti, 1977, pp. 82–84), as in the various versions of the traditional theory of value, i.e. classical (Ricardo, 1951, p. 46), Marxian (Marx, [1894] 1959, Chaps. 11 and 12), Austrian (Böhm-Bawerk, [1889] 1959, Vol. 2, pp. 86 and 356–358; von Weizsäcker, 1977), and neoclassical (see, e.g. Kemp, 1973; Stolper and Samuelson, 1941).

However, as Sraffa (1960) pointed out, leaving aside these two restrictive cases, changes in income distribution can activate complex capital revaluation effects, which imply that the direction of relative price movement is governed not only by the differences in the relevant capital intensities but also by the movement of the relevant capital intensities ('price effect') arising from changes in relative commodity prices:

> [T]he means of production of an industry are themselves the product of one or more industries which may in their turn employ a still lower proportion of labor to means of production (and the same may be the case with these latter means of production; and so on) (pp. 14–15). [...] [A]s the wages fall the price of the product of a low-proportion [...] industry may rise or it may fall, or it may even alternate in rising and falling, relative to its means of production (p. 15). [...] The reversals in the direction of the movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with *any* notion of capital as measurable quantity independent of distribution and prices. (p. 38; Sraffa, 1960)

[8] The largest ratio between two consecutive singular values of $\mathbf{K}$ provides a measure of the distance of a controllable pair $[\mathbf{J}, \mathbf{p}^{\mathrm{T}}(0)]$ to the nearest uncontrollable pair or, in other words, the order of perturbation needed to transform a controllable system into an uncontrollable one (Boley and Lu, 1986; for a comprehensive review of alternative approaches, see Datta, 2004, pp. 183–191).

Indeed, differentiation of equation (4) with respect to ρ finally gives (for a detailed analysis, see Mariolis and Tsoulfidis, 2016a, pp. 40–45)

$$\dot{p}_j \equiv dp_j/d\rho = Rv_j\left[\left(\kappa_j - R^{-1}\right) + \rho\dot{\kappa}_j\right]$$

where the difference $\kappa_j - R^{-1}$ denotes the traditional or capital-intensity effect, $\kappa_j \equiv (\mathbf{p}^{\mathrm{T}}\mathbf{H}\mathbf{e}_j)v_j^{-1}$ the capital-intensity of the vertically integrated industry producing commodity j, R^{-1} the capital-intensity of the Sraffian Standard system, which is independent of prices and income distribution (since, in our case, Sraffa's Standard commodity is the *numéraire*), and $\dot{\kappa}_j = (\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{H}\mathbf{e}_j)v_j^{-1}$ the Sraffian or price effect, which depends on the entire economic system and, therefore, is not predictable at the level of any single industry. Hence, when these two effects have opposite signs, i.e. $\kappa_j - R^{-1} > (<)\ 0$ and $\dot{\kappa}_j < (>)\ 0$, the traditional statement about the direction of relative price movements does not necessarily hold true, while the underlying phenomena call for a new approach to the theory of value and, therefore, form the basis of the Sraffian theory of value. In effect, all statements and relationships derived from the traditional theory of value framework cannot, in general, be extended beyond a world where: (i) there are no produced means of production; or (ii) there are produced means of production, while the profit rate on the value of those means of production is zero; or, finally, (iii) that profit rate is positive, while the economy produces one and only one, single or composite, commodity (Garegnani, 1970, 1984; Salvadori and Steedman, 1985). Consequently, it can be stated that the failures of the traditional theory of value arise from the existence of complex inter-industry linkages in the realistic case of production of commodities *and* positive profits by means of commodities. In a note written on 16 January 1946, Piero Sraffa remarked that

> if the "*Labour Theory of Value*" applied exactly throughout, then, and only then, would the "*marginal product of capital*" theory work! (Sraffa Papers D3/12/16: 34; cited in Kurz, 1998, p. 447)

II. I.d A spectral reconstruction of the theory of value

Leaving aside the aforementioned two unrealistic or restrictive cases (i.e. $\mathbf{v}^{\mathrm{T}}\mathbf{J} = \mathbf{v}^{\mathrm{T}}$, and $n = 2$), equations (4) through (7) imply that, from a theory of value viewpoint, it suffices to focus on the following seven ideal-type (in the Weberian sense) cases:[9]

Case 1: The economy *tends* to be decomposed into n quasi-similar self-reproducing vertically integrated industries, i.e. $\mathbf{J} \approx \mathbf{I}$ (consider Hartfiel and Meyer, 1998). It

[9] The first five cases have been extensively analyzed in the literature: Mariolis (2013, 2015); Mariolis and Tsoulfidis (2009, 2016a, pp. 154–157; 2018). Thus, here we report, without detailed proofs, the main findings that are directly relevant for our present purposes. To the best of my knowledge, the other two cases have not been addressed in the literature. Examples illustrating these two cases will appear in the forthcoming book by Mariolis et al. (2021, Chap. 2, Appendix).

follows that $\lambda_{\mathbf{J}k} \approx 1$ and $\mathbf{p}^{\mathrm{T}} \approx \mathbf{p}^{\mathrm{T}}(0)$. Hence, the economy tends to behave as a one-industry economy, and the labor theory of value tends to hold true. When $\mathbf{J} = \mathbf{I}$, the economy is controllable of rank 1, irrespective of the direction of the labor value vector, $\mathbf{p}^{\mathrm{T}}(0)$.

Case 2: There are strong quasi-linear dependencies amongst the technical conditions of production in all the vertically integrated industries, i.e. $|\lambda_{\mathbf{J}k}| \approx 0$ or $\mathbf{J} \approx (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1})^{-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$. It follows that

$$\mathbf{p}^{\mathrm{T}} \approx (1-\rho)\,\mathbf{p}^{\mathrm{T}}(0) + \rho\mathbf{p}^{\mathrm{T}}(1)$$

namely, $\mathbf{p}^{\mathrm{T}}$ tends to be a convex combination of the extreme, economically significant, values of the price vector, $\mathbf{p}^{\mathrm{T}}(0)$ and $\mathbf{p}^{\mathrm{T}}(1)$, and, therefore, linear.[10] When $|\lambda_{\mathbf{J}k}| = 0$, we obtain a 'rank-one economy,' i.e. rank[$\mathbf{J}$] $= 1$ or $\mathbf{J} = (\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1})^{-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$, which exhibits the following two essential characteristics: (i) Irrespective of the direction of $\mathbf{p}^{\mathrm{T}}(0)$, it holds that

$$\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{h} = \left[(1-\lambda_{\mathbf{A}1})\,\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}\right]^{-1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(1), h = 1, 2\ldots$$

since

$$\mathbf{J}^{h} = \left(\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}\right)^{-h}\left(\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{x}_{\mathbf{J}1}\right)^{h-1}\mathbf{x}_{\mathbf{J}1}\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}} = \mathbf{J}$$

Hence, rank[$\mathbf{K}$] $= 2$ and, therefore, the economy is uncontrollable, i.e. controllable of rank 2. (ii) It is equivalent, via Schur's triangularization (see, e.g. Meyer, 2001, pp. 508–509), to an *economically* significant and generalized $(1 \times n-1)$ Marx–Fel'dman–Mahalanobis (or, in more traditional terms, 'corn–tractor') economy (Mariolis, 2013; pp. 5195–5196, 2015, p. 270). Hence, it behaves as a reducible two-industry economy without 'self-reproducing non-basics' (in the sense of Sraffa, 1960, Appendix B). Consequently, on the one hand, the price side of a rank-one economy is 'a little' more complex than that of the labor theory of value economy ($\mathbf{J} = \mathbf{I}$) and, at the same time, much simpler than that of a completely controllable economy. In fact, its price side corresponds to that of the traditional theory of value. On the other hand, a rank-one economy can be fully described by a triangular matrix with only n positive technical coefficients and, therefore, its production structure is 'a little' more complex than that of Austrian-type economies, where the technical coefficients matrix is, by assumption, strictly triangular (see Burmeister, 1974).

Case 3: Consider the following rank-one perturbation of the labor theory of value economy (see Case 1): $\mathbf{J} \approx (1 + \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi})^{-1}[\mathbf{I} + \boldsymbol{\chi}\boldsymbol{\psi}^{\mathrm{T}}]$ (≥ 0). It follows that $\lambda_{\mathbf{J}k} \approx$

[10] It may be noted that, if the elements of $\mathbf{J}$ are identically and independently distributed, then Bródy's (1997) conjecture implies that $|\lambda_{\mathbf{J}2}|$ tends to zero, with speed $n^{-0.5}$, when n tends to *infinity* (see Mariolis and Tsoulfidis, 2016a, Chap. 6, and the references therein).

$(1+\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi})^{-1}$ (consider, e.g. Ding and Zhou, 2007, p. 1224) and

$$\mathbf{p}^{\mathrm{T}} \approx \left(1-\rho\lambda_{\mathbf{J}k}\right)^{-1}\left[(1-\rho)\,\mathbf{p}^{\mathrm{T}}(0)+\rho\left(1-\lambda_{\mathbf{J}k}\right)\mathbf{p}^{\mathrm{T}}(1)\right]$$

namely, $p_j(\rho)$ tend to be rational functions of degree 1 and, therefore, monotonic. Hence, for $-\infty << \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} << 0$ or $0 << \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} << +\infty$, the economy tends to behave as a two-industry economy with only basic commodities, and the traditional theory of value tends to hold true. As $\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} \to 0$ (as $\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\chi} \to \pm\infty$), we obtain Case 1 (Case 2).

Case 4: Consider the following rank-two perturbation of the labor theory of value economy, i.e. $\mathbf{J} \approx (1+\lambda_{\boldsymbol{\Psi}1})^{-1}[\mathbf{I}+\sum_{\kappa=1}^{2}\boldsymbol{\chi}_\kappa\boldsymbol{\psi}_\kappa^{\mathrm{T}}]$, where $\boldsymbol{\chi}_\kappa$, $\boldsymbol{\psi}_\kappa^{\mathrm{T}}$, are semi-positive vectors (or two pairs of complex conjugate vectors), and $\boldsymbol{\Psi} \equiv [\boldsymbol{\psi}_1,\boldsymbol{\psi}_2]^{\mathrm{T}}[\boldsymbol{\chi}_1,\boldsymbol{\chi}_2]$ (in either case, $\boldsymbol{\Psi}$ is a 2×2 matrix with only real eigenvalues). It follows that $n-2$ non-dominant eigenvalues of $\mathbf{J}$ tend to equal $(1+\lambda_{\boldsymbol{\Psi}1})^{-1}$, and the remaining tends to equal $(1+\lambda_{\boldsymbol{\Psi}2})(1+\lambda_{\boldsymbol{\Psi}1})^{-1}$ (consider Ding and Zhou, 2008, p. 635). Hence, the economy tends to behave as a three-industry economy; and the same holds true when $\lambda_{\mathbf{J}k} \approx \alpha_k \pm i\beta_k$, where $i \equiv \sqrt{-1}$ and $0 << |\beta_k|$, for all k.[11]

Case 5: The sub-dominant eigenvalues are complex, $\lambda_{\mathbf{J}2,3} \approx \alpha_{2,3} \pm i\beta_{2,3}$, where $0 << |\beta_{2,3}|$, and $\lambda_{\mathbf{J}4} \approx \cdots \approx \lambda_{\mathbf{J}n} \approx 0$. Hence, the economy tends to behave as a reducible four-industry economy without self-reproducing non-basics. Both in Cases 4 and 5, the price functions *may* be non-monotonic.

Case 6: Matrix $\mathbf{J}$ is *doubly* stochastic, i.e. $\mathbf{e}^{\mathrm{T}}\mathbf{J} = \mathbf{e}^{\mathrm{T}}$ and $\mathbf{J}\mathbf{e} = \mathbf{e}$. From equation (6) it follows that

$$\mathbf{p}^{\mathrm{T}}(1) = [(1-\lambda_{\mathbf{A}1})\,n]^{-1}\left(\mathbf{l}^{\mathrm{T}}\mathbf{e}\right)\mathbf{e}^{\mathrm{T}}$$

or, since $\mathbf{v}^{\mathrm{T}}\mathbf{e} = (1-\lambda_{\mathbf{A}1})^{-1}(\mathbf{l}^{\mathrm{T}}\mathbf{e})$ and $\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}}$,

$$\mathbf{p}^{\mathrm{T}}(1) = \overline{p}(0)\mathbf{e}^{\mathrm{T}} \tag{10}$$

where $\overline{p}(0) \equiv n^{-1}(\mathbf{p}^{\mathrm{T}}(0)\mathbf{e})$ equals the arithmetic mean of the elements of the labor value vector. Hence, if there is a commodity whose labor value equals the arithmetic mean of the labor values, then, by Rolle's theorem, its price curve will necessarily have at least one stationary point in the economically relevant interval of the profit rate.

Case 7: Since $\mathbf{A} = [\mathbf{I}+\mathbf{H}]^{-1}\mathbf{H}$, there is no good economic reason for supposing that $\mathbf{J}$ is doubly stochastic. It should be noted, however, that: (i) Any doubly stochastic matrix can be expressed as a convex combination of at most $(n-1)^2+1$ permutation matrices (see, e.g. Minc, 1988, pp. 117–122). (ii) Matrix $\mathbf{J}$ is similar to the *column*

[11] Any complex number is an eigenvalue of a positive 3×3 circulant matrix (Minc, 1988, p. 167). For the properties of the circulant matrices, see Davis (1979).

stochastic matrix $\mathbf{M} \equiv \hat{\mathbf{y}}_{\mathbf{J}1}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1}$:

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1} = \mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1} = \mathbf{e}^{\mathrm{T}}$$

The elements of $\mathbf{M}$ are independent of both the choice of physical measurement units and the normalization of $\mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$. Matrix $\mathbf{M}$ can be conceived of as a matrix of the relative shares of the capital goods in the cost of outputs, evaluated at $\rho = 1$, or, alternatively, as derived from $\mathbf{J}$ by changing the units in which the various commodity quantities are measured.[12] Moreover, the Dmitriev and Dynkin (1946) and Karpelevich (1951) inequalities for stochastic matrices imply that

$$\alpha_k + |\beta_k| \tan\left(\pi n^{-1}\right) \leq 1 \tag{11}$$

for each eigenvalue $\lambda_{\mathbf{M}k}$ $(= \lambda_{\mathbf{J}k}) = \alpha_k \pm i\beta_k$ (also see Swift, 1972). (iii) Finally, when there is only one commodity input in each industry (i.e. industry $\kappa, \kappa = 1, 2, \ldots, n-1$, produces the input for industry $\kappa + 1$, and industry n produces the input for industry 1), $\mathbf{A}$ is imprimitive or 'cyclic' (also see Schefold, 2008, pp. 8–14; Solow, 1952, pp. 35–36 and 40–41). Therefore, $\mathbf{M}$ is circulant and doubly stochastic (see Mariolis and Tsoulfidis, 2016a, pp. 165–167).

Thus, hereafter, we consider a '*basic* circulant' perturbation of the labor theory of value economy, i.e.

$$\mathbf{J} \approx \mathbf{C} \equiv c\mathbf{I} + (1 - c)\,\mathbf{\Pi}$$

where $0 \leq c < 1$, $\mathbf{\Pi} \equiv \mathrm{circ}[0, 1, 0, \ldots, 0]$ is the basic circulant permutation (or shift) matrix (post-multiplying any matrix by $\mathbf{\Pi}$ shifts its columns one place to the right), and $\mathbf{\Pi}^n = \mathbf{I}$. The eigenvalues of the circulant doubly stochastic matrix $\mathbf{C}$ are $c + (1 - c)\theta^{\kappa}$, where $\kappa = 0, 1, \ldots, n - 1$, $\theta \equiv \exp(2\pi i n^{-1})$, and

$$\theta^{\kappa} = \cos\left(2\pi\kappa n^{-1}\right) + i \sin\left(2\pi\kappa n^{-1}\right)$$

are the n distinct roots of unity. It then follows that: (i) The eigenvalues of $\mathbf{C}$ are the vertices of a regular n–gon, and $\mathbf{C}$ is that stochastic matrix that has an 'extremal eigenvalue' on the segment joining the points 1 and θ (Dmitriev and Dynkin, 1946; Karplevich, 1951).[13] This eigenvalue is a sub-dominant eigenvalue, which satisfies relation (11) as an *equality*. (ii) For $0 < c < 1$, the moduli of the eigenvalues of $\mathbf{C}$

[12] When rank[$\mathbf{J}$] = 1, all the columns of $\mathbf{M}$ are equal to each other (Mariolis and Tsoulfidis, 2009, p. 10; Iliadi et al., 2014, p. 40).

[13] A number λ is called extremal eigenvalue if (i) it belongs to the set of eigenvalues of a stochastic matrix; and (ii) $\alpha\lambda$ does not belong to this set, whenever $\alpha > 1$.

are given by

$$\sqrt{c^2 + 2c\,(1-c)\cos\left(2\pi\kappa n^{-1}\right) + (1-c)^2}$$

or, equivalently,

$$\sqrt{1 + 2c\,(1-c)\left[\cos\left(2\pi\kappa n^{-1}\right) - 1\right]}$$

which is a symmetric function with respect to $c = 0.5$ and κ', κ'', where $\kappa' + \kappa'' = n$ (Davis, 1979, pp. 119–120). The modulus of the sub-dominant eigenvalues occurs for $\kappa = 1, n-1$. When n is even, i.e. $n = 2\mu$, the smallest modulus occurs for $\kappa = \mu$, and equals $|1-2c|$, while when n is odd, $n = 2\mu + 1$, the smallest modulus occurs for $\kappa = \mu, \mu + 1$. Finally, $\mathbf{C}$ has rank $n-1$ iff n is even and $c = 0.5$ (Davis, 1979, p. 147). For instance, Figure 1 displays the location of the eigenvalues of $\mathbf{C}$ in the complex plane, for $n = 3, 6$ and $c = 0, 0.25, 0.75$.

Now, we turn to the price side of the economies $[\mathbf{C}, \mathbf{p}^{\mathrm{T}}(0)]$, $0 \le c < 1$. Ignoring the approximation error, equation (4) reduces to

$$\mathbf{p}^{\mathrm{T}} = (1-\gamma)\,\mathbf{p}^{\mathrm{T}}(0) + \gamma\mathbf{p}^{\mathrm{T}}\mathbf{\Pi} \tag{12}$$

where $\gamma \equiv (1-c)\rho(1-\rho c)^{-1}$, $0 \le \gamma \le 1$. Hence, since $\mathbf{\Pi}^n = \mathbf{I}$, equation (5) reduces to

$$\mathbf{p}^{\mathrm{T}} = (1-\gamma)\left(1-\gamma^n\right)^{-1}\mathbf{p}^{\mathrm{T}}(0)\left[\mathbf{I} + \gamma\mathbf{\Pi} + \gamma^2\mathbf{\Pi}^2 + \cdots + \gamma^{n-1}\mathbf{\Pi}^{n-1}\right]$$

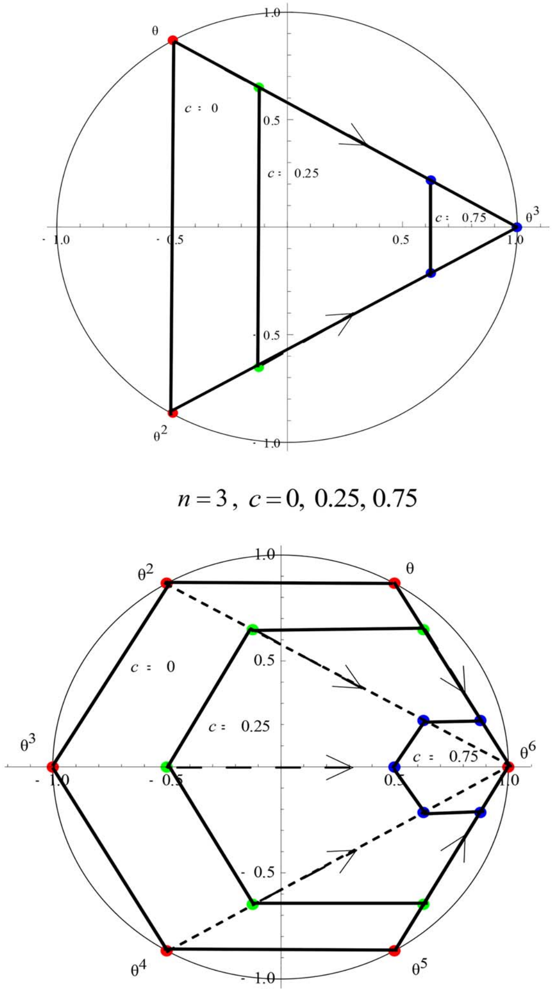
or, since $(1-\gamma)(1-\gamma^n)^{-1} = (1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1})^{-1}$,

$$\mathbf{p}^{\mathrm{T}} = \left(1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}\right)^{-1}\mathbf{p}^{\mathrm{T}}(0)\left[\mathbf{I} + \gamma\mathbf{\Pi} + \gamma^2\mathbf{\Pi}^2 + \cdots + \gamma^{n-1}\mathbf{\Pi}^{n-1}\right] \tag{13}$$

From equations (12) and (13) it follows that:

(i) Although matrix $\mathbf{C}$ is irreducible, commodity prices reduce to a *finite* series of dated quantities of vertically integrated labor. Hence, it can be stated that these 'basic circulant economies' bear some characteristic similarities with the 'old wine–oak chest' economy example constructed by Sraffa (1960, pp. 37–38). That example

> is a crucial test for the [traditional] ideas of a quantity of capital and of [an average] period of production. [...] One can only wonder what is the good of a quantity of capital or a period of production which, since it depends on the rate of interest [the rate of profits], cannot be used for its traditional purpose, which is to determine the rate of interest [the rate of profits]. (Sraffa, 1962, pp. 478–479)



$n=3,\ c=0,\ 0.25,\ 0.75$

$n=6,\ c=0,\ 0.25,\ 0.75$

FIGURE 1. The location of the eigenvalues of $\mathbf{C}$ in the complex plane; $n=3$, 6 and $c=0, 0.25, 0.75$.

(ii) In fact, because of the structure of the economies' matrices, commodity prices are governed by the terms

$$\delta_\kappa \equiv \left(1+\gamma+\gamma^2+\cdots+\gamma^{n-1}\right)^{-1}\gamma^\kappa, \kappa = 0, 1, \ldots, n-1$$

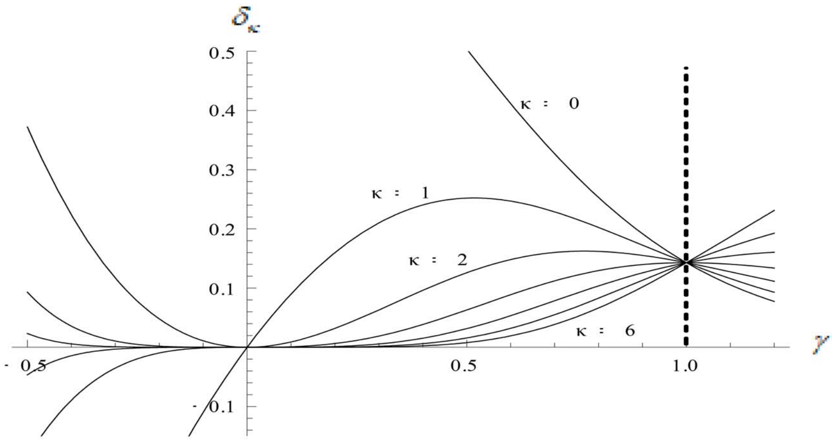


FIGURE 2. The rational function terms that govern the commodity prices in basic circulant economies; $n=7$, $-0.5 \leq \gamma \leq 1.2$.

where the denominator has either no real roots (when n is odd) or one real root (i.e. -1, when n is even). The first derivative of δ_κ with respect to γ is

$$\dot{\delta}_\kappa = \left(1 - \gamma^n\right)^{-2} \gamma^{\kappa-1} \left(\varepsilon_\kappa + \zeta_\kappa\right)$$

where $\varepsilon_\kappa \equiv \kappa - (1+\kappa)\gamma$ defines a linear function, and $\zeta_\kappa \equiv [(n-\kappa) - (n-\kappa-1)\gamma]\gamma^n$ defines a polynomial function. Hence, we get $\dot{\delta}_0(0) = -1$, $\dot{\delta}_1(0) = 1$ and $\dot{\delta}_\kappa(0) = 0$, for $\kappa \geq 2$, while $\dot{\delta}_\kappa(1) = (2n)^{-1}(1 + 2\kappa - n)$. Moreover, when $\kappa \geq 2$ is even (is odd), δ_κ has a minimum (an inflection point) at $\gamma = 0$. Finally, iff $1 \leq \kappa < 2^{-1}(n-1)$ and $n \geq 4$, then the equation $\varepsilon_\kappa + \zeta_\kappa = 0$ has two roots in the interval $[0, 1]$, i.e. $\gamma = \gamma_\kappa^*$ (unique), where $0 < \gamma_\kappa^* < 1$, at which δ_κ is maximized, and $\gamma = 1$ (repeated), where $\dot{\varepsilon}_\kappa(1) + \dot{\zeta}_\kappa(1) = 0$ (in all other cases, it has, in the said interval, the roots 0 and/or 1). For instance, Figure 2 displays the terms δ_κ as functions of γ, for $n = 7$: $\gamma_1^* \cong 0.517$, $\gamma_2^* \cong 0.768$, and δ_3 has a maximum at $\gamma = 1$. The values γ_κ^* tend to the values of the sequence $(1+\kappa)^{-1}\kappa$ as n tends to infinity and, therefore, the maximum values of δ_κ tend to the values of the sequence $(1+\kappa)^{-(1+\kappa)}\kappa^\kappa$.

(iii) Commodity prices tend to $\mathbf{p}^{\mathrm{T}}(0)\mathbf{\Pi}^{n-1}$ as γ tends to plus or minus infinity. Iff there exists a non-zero value of γ, say γ^{**}, such that $p_j(\gamma^{**}) = p_j(0)$, then

$$p_j\left(\gamma^{**}\right) = \mathbf{p}^{\mathrm{T}}\left(\gamma^{**}\right)\mathbf{\Pi}\mathbf{e}_j = p_{j-1}\left(\gamma^{**}\right)$$

where $j = 1, 2, \ldots, n$ and $p_0(\gamma^{**}) \equiv p_n(\gamma^{**})$.

(iv) Differentiation of equation (12) with respect to ρ gives

$$\dot{\mathbf{p}}^{\mathrm{T}} = -\dot{\gamma}\left(\mathbf{p}^{\mathrm{T}}(0) - \mathbf{p}^{\mathrm{T}}\mathbf{\Pi}\right) + \gamma\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{\Pi} \tag{14}$$

$$\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{e} = 0 \tag{15}$$

where $\dot{\gamma} \equiv (1-c)(1-\rho c)^{-2} > 0$, the difference $\mathbf{p}^{\mathrm{T}}\mathbf{\Pi} - \mathbf{p}^{\mathrm{T}}(0)$ represents the capital-intensity effect, while the term $\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{\Pi}$ represents the price effect. Now, it suffices to focus on the extreme, economically significant, values of ρ: (a). At $\rho = 0$ equation (14) reduces to

$$\dot{\mathbf{p}}^{\mathrm{T}}(0) = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{D} \tag{16}$$

where $\mathbf{D} \equiv \mathbf{I} - \mathbf{\Pi}$ is a circulant double-centered matrix, since all its columns and rows sum to zero, i.e. $\mathbf{e}^{\mathrm{T}}\mathbf{D} = \mathbf{0}^{\mathrm{T}}$, $\mathbf{De} = \mathbf{0}$, and rank$[\mathbf{D}] = n-1$. (b). At $\rho = 1$ equation (14) reduces to

$$\dot{\mathbf{p}}^{\mathrm{T}}(1) = -(1-c)^{-1}\left(\mathbf{p}^{\mathrm{T}}(0) - \mathbf{p}^{\mathrm{T}}(1)\mathbf{\Pi}\right) + \dot{\mathbf{p}}^{\mathrm{T}}(1)\mathbf{\Pi}$$

or, rearranging terms and invoking equations (10) and $\mathbf{e}^{\mathrm{T}}\mathbf{\Pi} = \mathbf{e}^{\mathrm{T}}$,

$$\dot{\mathbf{p}}^{\mathrm{T}}(1)\mathbf{D} = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{F} \tag{17}$$

where $\mathbf{F} \equiv \mathbf{I} - n^{-1}(\mathbf{ee}^{\mathrm{T}})$ is the centering matrix, which is symmetric and idempotent (multiplication of any vector by the centering matrix has the effect of subtracting its arithmetic mean from every element). The solution to equations (15) and (17) is given by

$$\dot{\mathbf{p}}^{\mathrm{T}}(1) = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{FD}^{+}$$

or

$$\dot{\mathbf{p}}^{\mathrm{T}}(1) = -(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{D}^{+} \tag{18}$$

where $\mathbf{D}^{+}$ denotes the Moore–Penrose inverse of $\mathbf{D}$, which is, in our case, a circulant double-centered matrix satisfying $\mathbf{DD}^{+} = \mathbf{D}^{+}\mathbf{D} = \mathbf{F}$.[14] Moreover, when n is even, $n = 2\mu$, the explicit expression for matrix $\mathbf{D}^{+}$ can be written as

$$\mathbf{D}^{+} = (4\mu)^{-1}\mathrm{circ}\left[2\mu-1, 2\mu-3, 2\mu-5, \ldots, -(2\mu-3), -(2\mu-1)\right] \tag{19}$$

[14] There is an algebraic analog of equations (15) and (17) in electrical network theory: $\dot{\mathbf{p}}^{\mathrm{T}}(1)$ and $-(1-c)^{-1}\mathbf{p}^{\mathrm{T}}(0)\mathbf{F}$ correspond to the vectors of voltages and currents, respectively; equations (15) and (17) correspond to Kirchhoff's voltage law and Ohm's law, respectively; $\mathbf{D}$ and $\mathbf{D}^{+}$ correspond to the matrices of admittance and impedance, respectively (see Sharpe and Styan, 1965).

while when n is odd, $n = 2\mu + 1$, it can be written as

$$\mathbf{D}^{+} = (2\mu + 1)^{-1}\text{circ}\,[\mu, \mu - 1, \mu - 2, \ldots, -(\mu - 1), -\mu] \tag{20}$$

(consider Davis, 1979, pp. 148–149). The elements of the first row of $-\mathbf{D}^{+}$ are equal to $\dot{\delta}_{\kappa}(1) = (2n)^{-1}(1 + 2\kappa - n)$, $\kappa = 0, 1, \ldots, n-1$. Hence, it is easy to check that equations (16), (18), (19), and (20) imply that, when $n \geq 3$ and $p_j(0) < p_{j+1}(0)$, $j = 1, 2, \ldots, n-1$, there is at least one element of $\dot{\mathbf{p}}^{\mathrm{T}}$, say $\dot{p}_h$, such that $\dot{p}_h(0)\dot{p}_h(1) < 0$, *irrespective* of the direction of $\mathbf{p}^{\mathrm{T}}(0)$. Then, by Bolzano's theorem, it follows that p_h necessarily has at least one extreme point in the interval $(0, 1)$.

Finally, it should be noted that $n \times n$ doubly stochastic circulant economies of the form

$$c_1\mathbf{I} + c_2\mathbf{\Pi} + c_2\mathbf{\Pi}^2 + \cdots + c_n\mathbf{\Pi}^{n-1}, (c_2, c_3, \ldots, c_{n-1}) > 0$$

do *not* necessarily generate non-monotonic price curves.

II. I.e The complex plane location of the polar theories of value

These seven ideal-type cases (and their possible combinations) indicate that the location of the non-dominant eigenvalues in the complex plane could be considered as an index for the underlying inter-industry linkages. More specifically, the analysis showed that, ignoring the approximation error, the hitherto alternative theories of value can be *algebraically* represented as 'perturbations' of the labor theory of value, i.e. of Case 1. Cases 2 and 3 correspond to the traditional theory of value, while Cases 4, 5, and 6 fall into the Sraffian theory of value. Finally, it could be said that Case 7, i.e. the basic circulant perturbation of the labor theory of value economy, corresponds to the 'Sraffian *polar* theory of value,' since in that case the price–profit rate relationship is non-monotonic whatever the labor value vector is. Hence, Figure 3 displays the location of the polar theories of value in the complex plane.

III. THE DEGREE AND NUMERICAL RANK OF PRICE CONTROLLABILITY OF ACTUAL ECONOMIES

The Sraffian price–wage–profit rate system of quite diverse actual economies (but, *ex hypothesis*, linear and single-product) or, to be more precise, of their Symmetric Input–Output Tables (SIOT) simulacra, has been examined in a relatively large number of studies. The key stylized findings were that, in the economically relevant interval of the profit rate:[15] (i) Non-monotonic price–profit rate curves *do* exist. Nevertheless, they are not significantly more than 20% of the tested cases, while, expressed in terms of Sraffa's Standard commodity, they have no more than one extreme point. Cases of reversal in the direction of deviation between prices and labor values ('price–labor

[15] See Mariolis and Tsoulfidis (2016a, Chaps. 3, 5, and 6, 2016b, 2018); Mariolis et al. (2019) and the references therein.

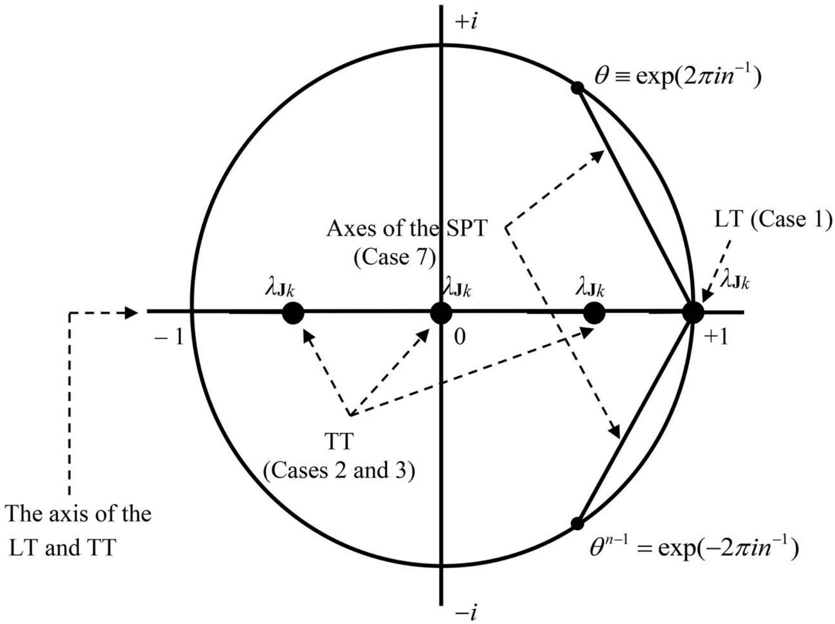


FIGURE 3. The complex plane location of the polar theories of value: labor theory (LT), traditional theory (TT), and Sraffian polar theory (SPT).

value reversals') also occur, but are rarer. (ii) Wage–profit rate curves with alternating curvature *do* exist. Nevertheless, despite the presence of considerable deviations from the 'equal value compositions of capital' case, the wage–profit rate curves are near-linear, in the sense that the correlation coefficients between the distributive variables tend to be above 99%, and their second derivatives change sign no more than once or, very rarely, twice, irrespective of the *numéraire* chosen. (iii) Therefore, the approximation of the empirical price–wage–profit rate curves through low-order formulae (ranging from linear to quadratic) works well. (iv) Since actual economies are characterized by complex inter-industry linkages, alternative production methods, and the production of many commodities and positive profits by means of many commodities, the aforementioned shapes of the price–wage–profit rate curves seem to be *paradoxical*. However, they can be explained by the fact that, across countries and over time, both the moduli of the first non-dominant eigenvalues and the first non-dominant singular values of matrices $\mathbf{J}$ fall quite rapidly, whereas the rest constellate in much lower values, forming 'long tails'. Hence, although rank$[\mathbf{J}] = n$ holds true, the 'effective rank (or dimensions)' of $\mathbf{J}$ is much lower than n.

The aforementioned stylized findings in combination with the theoretical analysis developed in this paper suggest that the actual single-product economies *tend* to behave as three-industry uncontrollable systems. To look deeper into this interesting and important phenomenon, we will deal with data from ten flow Symmetric Input–Output Tables (SIOTs) of five European economies, i.e. Denmark (for the years 2000 and 2004; $n = 56$), Finland (for the years 1995 and 2004; $n = 57$), France (for the years 1995, $n = 58$, and 2005, $n = 57$), Germany (for the years 2000 and 2002; $n = 57$) and Sweden (for the years 1995, $n = 53$, and 2005, $n = 51$). These SIOTs have been firstly used by Iliadi *et al*. (2014), and their findings (for instance, non-monotonic price curves, expressed in terms of Sraffa's Standard commodity, are observed in about 105/559 or 19% of the tested cases) are absolutely consistent with those of *all* other studies of actual price–wage–profit rate systems. Hence, this data sample could be considered as sufficiently representative. Table 1 reports: (i) $|\lambda_{\mathbf{J}2}|$, $|\lambda_{\mathbf{J}3}|$, $|\lambda_{\mathbf{J}n}|$ and the geometric mean, *GM*, of the moduli of the non-dominant eigenvalues of $\mathbf{J}$ (reproduced from Iliadi *et al*., 2014, p. 43), which can be written, in our case, as

$$GM = |\det[\mathbf{J}]|^{(n-1)^{-1}} = \left(\prod_{i=1}^{n} \sigma_{\mathbf{J}i}\right)^{(n-1)^{-1}}$$

As is well known, the geometric mean is rather appropriate for detecting the central tendency of an exponential set of numbers. (ii) The ratio between the smallest and the largest singular values, $\sigma_{\mathbf{J}n}\sigma_{\mathbf{J}1}^{-1}$, of $\mathbf{J}$. (iii) The absolute values of the determinant of the Krylov–controllability matrices and of the determinant of the Vandermonde matrices of the eigenvalues of $\mathbf{J}$ (see equation (8)). (iv) The numbers of non-monotonic price–profit rate curves and of price–labor value reversals, denoted by 'N-M' and 'Rev.', respectively. It then follows that, in total, there are 63/559 or 11% cases of price–labor value reversals. (v) The degree of price controllability, *DC* (see equation (9)). (vi) The 'relative or normalized numerical rank of price controllability', *NNRC*, defined as

$$NNRC(\overline{\tau}) \equiv 100n^{-1}NR(\mathbf{K}, \overline{\tau})$$

where $NR(\mathbf{K}, \overline{\tau})$ denotes the numerical rank of $\mathbf{K}$, i.e. the number of singular values of $\mathbf{K}$ that are larger than $\overline{\tau}\sigma_{\mathbf{K}1}$, and $\overline{\tau}$ denotes the chosen level of tolerance. Finally, Figure 4 (reproduced from Iliadi *et al*., 2014, p. 45) displays the location of the eigenvalues of all matrices $\mathbf{J}$ in the complex plane, while Figure 5 (the horizontal axis is plotted in logarithmic scale) displays the normalized singular values, $\sigma_{\mathbf{K}j}\sigma_{\mathbf{K}1}^{-1}$, of all matrices $\mathbf{K}$.

From these representative results, it is deduced that: (i) Non-monotonic price–profit rate curves appear when the price and the capital-intensity effects work in opposite directions, and the former dominate the latter. Moreover, the price–labor value reversals (a) imply that the identification of a vertically integrated industry as 'labor (or capital)-intensive' makes sense only with respect to a given profit rate,

TABLE 1.
Spectral and price controllability characteristics of actual economies: five European economies, 10 symmetric input–output tables

	Denmark		Finland		France		Germany		Sweden	
	2000 $n=56$	2004 $n=56$	1995 $n=57$	2004 $n=57$	1995 $n=58$	2005 $n=57$	2000 $n=57$	2002 $n=57$	1995 $n=53$	2005 $n=51$
$\lvert\lambda_{\mathbf{J}2}\rvert$	0.522	0.638	0.597	0.850	0.611	0.588	0.570	0.610	0.532	0.422
$\lvert\lambda_{\mathbf{J}3}\rvert$	0.486	0.502	0.433	0.503	0.529	0.453	0.497	0.516	0.434	0.390
$\lvert\lambda_{\mathbf{J}n}\rvert$	6×10^{-4}	1×10^{-3}	1×10^{-3}	3×10^{-4}	3×10^{-5}	1×10^{-3}	1×10^{-3}	6×10^{-3}	3×10^{-3}	9×10^{-4}
GM	0.07	0.07	0.05	0.05	0.06	0.08	0.11	0.11	0.05	0.05
$\sigma_{\mathbf{J}n}\sigma_{\mathbf{J}1}^{-1}$	9×10^{-5}	1×10^{-3}	1×10^{-3}	5×10^{-5}	10^{-5}	2×10^{-4}	9×10^{-5}	4×10^{-4}	5×10^{-4}	10^{-4}
N-M	13 (23%)	18 (32%)	9 (16%)	13 (23%)	11 (19%)	8 (14%)	9 (16%)	7 (12%)	11 (21%)	6 12%)
Rev.	5 (9%)	13 (23%)	6 (11%)	13 (23%)	6 (10%)	2 (4%)	7 (12%)	3 (5%)	5 (9%)	3 (6%)
$\lvert\det[\mathbf{K}]\rvert$	10^{-782}	10^{-793}	10^{-727}	10^{-740}	10^{-741}	4×10^{-744}	10^{-728}	10^{-722}	2×10^{-727}	6×10^{-721}
$\lvert\det[\mathbf{V}]\rvert$	2×10^{-647}	3×10^{-657}	3×10^{-674}	2×10^{-674}	3×10^{-681}	2×10^{-672}	1×10^{-659}	4×10^{-646}	3×10^{-610}	3×10^{-591}
DC	6×10^{-19}	6×10^{-19}	8×10^{-19}	1×10^{-19}	5×10^{-19}	8×10^{-19}	9×10^{-20}	6×10^{-19}	5×10^{-19}	3×10^{-20}
NNRC(10^{-4})	9%	11%	11%	11%	10%	11%	11%	11%	9%	10%
NNRC(10^{-2})	5%	5%	5%	7%	5%	5%	5%	5%	6%	6%

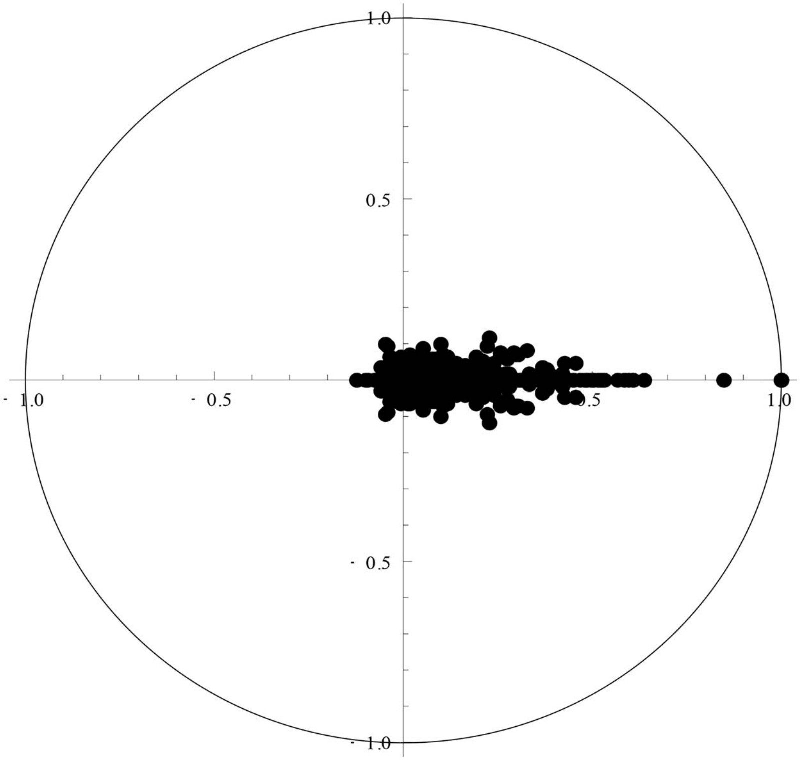


FIGURE 4. The complex plane location of the eigenvalues of all normalized vertically integrated technical coefficients matrices: five European economies, 10 symmetric input–output tables.

while, as Eatwell (2019) emphasizes, there is no neoclassical theory of the profit rate; and (b) are *analogous* to the re-switching of techniques phenomenon[16] and, therefore, indicate that there is no reason to consider that the empirical probability of this phenomenon is negligible. Since actual economies exhibit non-monotonic price–profit rate curves and price–labor value reversals, they cannot be coherently analyzed in terms of the traditional theory of value. (ii) It need hardly be said that the existence of fairly good, low-order approximations to the empirical price–profit rate curves is insufficient to restore the traditional theory of value. Therefore, only the Sraffian theory of value provides a sound empirical basis, although the eigenvalue distributions of the actual matrices **J** sharply differ from those of the basic circulant economies, which correspond to the Sraffian polar theory of value (compare Figure 1 with Figure 4). In fact, the actual eigenvalue distributions can be viewed as mixed

[16] 'Merge' the two numerical examples provided by Pasinetti (1966, pp. 504–508) and Sraffa (1960, pp. 37–38), and take into account Sraffa's (1960, pp. 81–82) relevant remark.

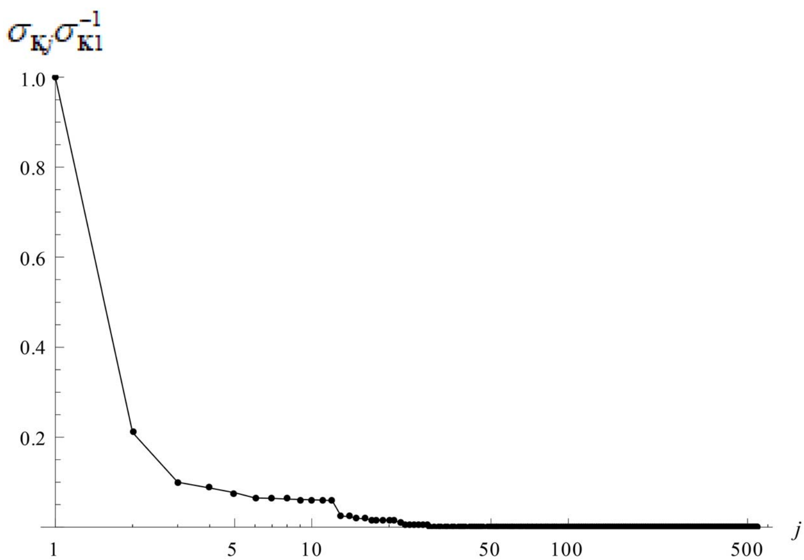


FIGURE 5. The normalized singular values of all price controllability matrices: five European economies, 10 symmetric input–output tables.

combinations of the ideal-type Cases 4 and 5 (presented in Section II.d). (iii) The actual single-product economies are characterized by rather low degrees and relatively low normalized numerical ranks of price controllability. This primarily results from the skew characteristic value distributions of the actual matrices $\mathbf{J}$, and indicates that the said economies constitute almost uncontrollable systems (see Table 1 and Figure 5).[17] In this connection, experiments with Krylov matrices formed from pseudo-random[18] vectors $\mathbf{p}^{\mathrm{T}}(0)$ and the abovementioned actual matrices $\mathbf{J}$ lead to similar results, i.e. to degrees of controllability of the order of 10^{-19}.

It should, finally, be added that, regarding actual Krylov matrices, we also experimented with the input–output data used by Soklis (2011), i.e. 10 Supply and Use Tables (SUTs) of the Finnish economy (for the years 1995 through 2004; $n = 57$), and the results were similar. As is well known, in the SUTs there are industries that produce more than one commodity, and commodities that are produced by more than one industry; therefore, the SUTs are the empirical counterpart of joint production economies *à la* von Neumann–Sraffa. For instance, when the Krylov matrix is formed from the vector $\mathbf{l}^{\mathrm{T}}[\mathbf{B}-\mathbf{A}]^{-1}$ and the matrix $\mathbf{A}[\mathbf{B}-\mathbf{A}]^{-1}$, where $\mathbf{B}$ denotes the output coefficients matrix, the degree of controllability is in the range of 6×10^{-93} to 10^{-27}, while, when the Krylov matrix is formed from $\mathbf{l}^{\mathrm{T}}\mathbf{B}^{-1}$ and $\mathbf{AB}^{-1}$, the degree of

[17] Mariolis and Veltsistas (2020) provide new, extensive empirical evidence (from 43 countries and 172 SIOTs, spanning the time period 2000–2014) that further supports this statement.

[18] Generated by *Mathematica*; see https://reference.wolfram.com/language/tutorial/PseudorandomNumbers.html

controllability is in the range of 2×10^{-28} to 2×10^{-20}. In those SUTs (i) there exists an interval of $r(> 0)$, such that the vector of 'labor-commanded' prices, $w^{-1}\mathbf{p}^{\mathrm{T}}$, is positive, for the years 1995 through 1998 and 2000 through 2002; and (ii) the monotonicity of the estimated wage–profit rate curves (for the years 1995, 1997, 2000, and 2001) depends on the numeraire chosen (Soklis, 2011, pp. 553–555). However, the existence of non-monotonic wage–profit rate curves contradicts the internal logic of the traditional theory of value and, at the same time, seriously undermines the construction of 'approximate surrogate production functions' from 'near-linear' wage–profit rate curves.

IV. CONCLUDING REMARKS

It has been shown that the hitherto competing theories of value (i) correspond to specific production structures and, therefore, to specific spectral characteristics of the price controllability matrix; and (ii) can be represented algebraically and, furthermore, understood conceptually as 'perturbations' of the so-called labor theory of value, which is a polar theory that holds true when, and only when, the rank of the price controllability matrix equals one. Thus, this paper pointed out a spectral reconstruction of the theory of value, which forms a typical, mathematical model for the most general theory of value to date, namely, the Sraffian theory, determined the location of the hitherto competing theories of value in the complex plane, and might provide a representation of the evolution of these theories in terms of *both* the logic and the history of economic thought.

It has also been shown that, although the existence of price–profit rate curves that are non-monotonic irrespective of the labor vector direction presupposes eigenvalue distributions sharply different from those appearing in actual economies, the Sraffian theory is not only the most general to date, but also empirically relevant. Since actual economies exhibit non-monotonic price–wage–profit rate curves, wage–profit rate curves with alternating curvature, and price–labor value reversals, which are analogous to the re-switching of techniques phenomenon, they can only be treated through the Sraffian theory.

At the same time, empirical evidence suggests that the actual economies are characterized by rather low degrees and relatively low normalized numerical ranks of price controllability. This finding results from the skew characteristic value distributions of the actual vertically integrated technical coefficients matrices, and indicates that the actual economies tend to respond as uncontrollable systems, with only a relatively few effectively controllable modes. Finally, this property of almost uncontrollability explains, in turn, the specific shape features of the empirical price–wage–profit rate curves that are (re-)positioned, by the traditional theory of value, at the heart of the capital theory debate.

Future research work should (i) expand the empirical analysis of the joint production economies using data from the Supply and Use Tables; (ii) delve into the proximate determinants of the uncontrollable aspects of real-world economies, and draw their broader implications for both political economy and economic policy issues; and (iii) heuristically look for eigenvalue locations in the complex plane that could lead to new versions of the theory of value.

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doi:10.1093/cpe/bzab004
Advance Access Publication 18 May 2021